

A mathematical–physical approach on regularity in hit-and-miss hypertopologies for fuzzy set multifunctions

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Abstract In this paper, an approach concerning hit-and-miss hypertopologies and especially regularity property viewed both as a continuity property in a hit-and-miss hypertopology (from a mathematical point of view) and also as a physical approximation property is intended.

Keywords Hit-and-miss hypertopologies · Hausdorff topology · Vietoris topology · Wijsman topology · Regularity · Approximations · Fractal theories · Non-differentiable physics · Scale relativity theory

Mathematics Subject Classification 28C15 · 49J53

Introduction

Nowadays, Hausdorff, Vietoris, Wijsman, Fell, Attouch-Wets, etc., hypertopologies are intensively studied due to their various applications in optimization, convex analysis, economics, image processing, sound analysis and synthesis (see Beer [7], Apreutesei [4], Hu and Papageorgiou [20], etc., concerning the Vietoris topology). Results involving the Hausdorff distance were obtained by Lorenzo and Maio [26] in melodic similarity, Lu et al. [27]—an approach to word image matching, etc. Recently, it was

shown that using proximity, all hypertopologies known so far are of the type hit-and-miss, which led to the unification of all hypertopologies under one topology called *the Bombay Hypertopology* [29].

The idea of modeling at multiple scales the phenomena behavior has become a useful tool in pure mathematics, applied mathematics physics and so on. Fractals are multiscale objects, which often describe such phenomena better than traditional mathematical models do. That is why fractal-based techniques lie at the heart of these areas. Kunze et al. [21] and Wicks [42] developed hyperspace theories concerning the Hausdorff metric and the Vietoris topology, as a foundation for self-similarity and fractality. In fact, for many years, topological methods were used in many fields to study the chaotic nature in dynamical systems (see for instance Sharma and Nagar [40], Wang et al. [41], Gómez-Rueda et al. [18], Li [24], Liu et al. [25], Ma et al. [28], Fu and Xing [11], etc.). These phenomena seem to be collective (set-valued), emerging out of many segregated components, having collective dynamics of many units of individual systems. This arose the need of a topological study of such collective dynamics. Recent studies of dynamical systems, in engineering and physical sciences, have revealed that the underlying dynamics is set-valued (collective), and not of a normal, individual kind, as it was usually studied before.

Also, the reader can refer to Lewin et al. [22] and Brown [8] for interesting approaches of topology in psychology. We also mention different aspects concerning generalized fractals in hyperspaces endowed with Hausdorff, or, more generally, with Vietoris hypertopology (see Andres and Fišer [2], Andres and Rypka [3], Banach and Novosad [5], Kunze et al. [21]).

Since in some examples of fractals (like neural networks and the circulatory system), the uniform property of the

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Hausdorff topology is inappropriate, we could intend to choose a convenient topology on the set of values of the studied multifunctions. In this sense, Wijsman topology may be preferred instead of Hausdorff topology because Wijsman topology could describe better the pointwise properties of fractals.

On the other hand, recently, domain theory has been studied in theoretical computer science, as a mathematical theory of semantics of programming languages (Edalat [10], Gierz et al. [17], etc.). In this context, (hyper-)topological notions from Mathematical Analysis as well as measure theory, dynamical systems or fractality can be considered via domain theory, obtaining computational models. Namely, in denotational semantics and domain theory, power domains are domains of nondeterministic and concurrent computations. As it is well-known, domain theory was introduced by Scott in theoretical computer science as a mathematical theory of semantics of programming languages.

Together with the increasing interest in hypertopologies, non-additive set multifunctions theories developed. In this context, regularity is known as an important continuity property with respect to different topologies, but, at the same time, it can be interpreted as an approximation property. Using regularity, we can approximate “unknown” sets by other sets which we have more informations. Usually, from a mathematical perspective, this approximation is done from the left by closed sets, or more restrictive, by compact sets and/or from the right by open sets. As a mathematical direct application of regularity, the classical Lusin’s theorem concerning the existence of continuous restrictions of measurable functions is very important and useful for discussing different kinds of approximation of measurable functions defined on special topological spaces and for numerous applications in the study of convergence of sequences of Sugeno and Choquet integrable functions (see Li et al. [23] for an interesting application of Lusin theorem), in the study of the approximation properties of neural networks, as the learning ability of a neural network is closely related to its approximating capabilities. Also, regular Borel measures are important tools in studies on the Kolmogorov fractal dimension (Barnsley [6], Mandelbrot [30], etc.). Lebesgue measure is a remarkable example of a regular measure.

The paper is organized as follows: in “[Hit-and-miss hypertopologies: an overview](#)” and “[Regular set multifunctions](#)” several remarkable hit-and-miss hypertopologies and their properties are listed from a mathematical perspective and regularity of set multifunctions is introduced in a unifying way with respect to these hypertopologies. In “[Regularization by sets of functions of \$\varepsilon\$ -approximation-type scale. Physical correspondences with hit-and-miss topologies](#)” and “[Conclusions](#)”, a physical

perspective concerning regularity and fractality is provided.

Our unifying mathematical–physical point of view on fractality, hypertopologies and regularity was initiated in our recent works [12, 13, 15, 16].

Hit-and-miss hypertopologies: an overview

Hausdorff, Vietoris and Wijsman, etc., topologies are remarkable examples of the so-called hit-and-miss hypertopologies. Like some physical concepts, these hypertopologies, although are composed of two independent parts, upper and lower hypertopologies, they become consistent when seen together. For instance, in physical terms, the non-differentiability of the curve motion of the physical object involves the simultaneous definition at any point of the curve, of two differentials (left and right). Since we cannot favor one of the two differentials, the only solution is to consider them simultaneously through a complex differential. Its application, multiplied by dt , where t is an affine parameter, to the field of space coordinates implies complex speed fields.

We used the following (selected) references: Apreutesei [4], Beer [7], Gavriluț and Apreutesei [14], Kunze et al. [21], Hu and Papageorgiou [20, Ch. 1], Precupanu et al. [38, Ch. 1], G. Apreutesei in Precupanu et al. [39, Ch. 8], Maio and Naimpally [29], etc.

We now briefly recall and list the definitions and main properties of the above-mentioned hypertopologies:

Vietoris topology

Let (X, τ) be a Hausdorff, topological space and $\mathcal{P}_0(X)$, the family of all nonvoid subsets of X . We consider

$M^- = \{C \in \mathcal{P}_0(X) / M \cap C \neq \emptyset\}$ (i.e., C hits M), $M^+ = \{C \in \mathcal{P}_0(X) / C \subseteq M\}$ (i.e., C misses cM), $S_{UV} = \{D^+ / D \in \tau\}$ and $S_{LV} = \{D^- / D \in \tau\}$.

Vietoris topology $\widehat{\tau}_V$ on $\mathcal{P}_0(X)$ has as a subbase the class $S_{UV} \cup S_{LV}$ and it is the supremum $\widehat{\tau}_V = \widehat{\tau}_V^+ \cup \widehat{\tau}_V^-$ of the lower and upper Vietoris topologies:

$\widehat{\tau}_V^+$ —the upper Vietoris topology ($\widehat{\tau}_V^-$ —the lower Vietoris topology, respectively) is the topology which has as a subbase the class S_{UV} (S_{LV} , respectively).

For $U, V \in \tau$, define $\mathcal{B}_{U, V_1, V_2, \dots, V_k} = U^+ \cap V_1^- \cap V_2^- \cap \dots \cap V_k^-$.

The family $\mathcal{B}_{U, V_1, V_2, \dots, V_k}$ of such subsets, where $U, V_1, V_2, \dots, V_k \in \tau$, is a base for the topology $\widehat{\tau}_V$ and the family of subsets $\mathcal{B}_U = U^+$ ($\mathcal{B}_V = V^-$, respectively) is a base for $\widehat{\tau}_V^+$ ($\widehat{\tau}_V^-$, respectively).

In different continuity properties (regularity for instance), the following observation is used:

Remark 2.1 [39, Ch. 8] A net $(A_i)_{i \in I} \subset \mathcal{P}_0(X)$ is:

- (i) $\widehat{\tau}_V^-$ -convergent to $A_0 \in \mathcal{P}_0(X)$ if for every $V \in \tau$, with $A_0 \cap V \neq \emptyset$, $\exists i_V \in I$ so that for every $i \in I, i \geq i_V$, we have $A_i \cap V \neq \emptyset$;
- (ii) $\widehat{\tau}_V^+$ -convergent to $A_0 \in \mathcal{P}_0(X)$ if for every $V \in \tau$, with $A_0 \subset V$, $\exists i_V \in I$ so that for every $i \in I, i \geq i_V$, we have $A_i \subset V$.

In what follows, let (X, d) be a metric space. By $\mathcal{P}_f(X)$ we mean the family of closed, nonvoid sets of X , by $\mathcal{P}_{bf}(X)$ the family of bounded, closed, nonvoid sets of X and by $\mathcal{P}_k(X)$, the family of all nonvoid compact subsets of X . τ_d denotes the topology induced by the metric d .

Wijsman topology

Wijsman topology τ_W on $\mathcal{P}_0(X)$ is the supremum of the *upper Wijsman topology* τ_W^+ and the *lower Wijsman topology* τ_W^- : $\tau_W = \tau_W^+ \cup \tau_W^-$.

The family

$$\mathcal{F} = \{M \in \mathcal{P}_0(X); d(x, M) < \varepsilon\}_{\varepsilon > 0} \cup \{M \in \mathcal{P}_0(X); d(x, M) > \varepsilon\}_{\varepsilon > 0}^{x \in X}$$

is a subbase for τ_W on $\mathcal{P}_0(X)$.

Let $M \in \mathcal{P}_0(X)$, $\{x_1, x_2, \dots, x_n\} \subset X, \varepsilon > 0$ be arbitrarily chosen.

τ_W^- (τ_W^+ , respectively) is generated by the family $U_W^-(M, x_1, x_2, \dots, x_n, \varepsilon) = \{N \in \mathcal{P}_0(X); d(x_i, N) < d(x_i, M) + \varepsilon, \text{ for every } i = \overline{1, n}\}$ ($U_W^+(M, x_1, x_2, \dots, x_n, \varepsilon) = \{N \in \mathcal{P}_0(X); d(x_i, M) < d(x_i, N) + \varepsilon, \text{ for every } i = \overline{1, n}\}$, respectively).

Proposition 2.2 (Apreutesei, Ch. 8 in Precupanu et al. [39]) $\tau_W^- = \tau_V^-$.

Remark 2.3

(I) Suppose $\{M_i\}_{i \in I} \subset \mathcal{P}_0(X)$. The following statements are equivalent:

- (i) $M_i \xrightarrow{\tau_W} M \in \mathcal{P}_0(X)$;
- (ii) For every $x \in X$, $d(x, M_i) \xrightarrow{p} d(x, M)$ (point-wise convergence);
- (iii) $M_i \xrightarrow{\tau_W^+} M$ and $M_i \xrightarrow{\tau_W^-} M$.

(II)

- (i) $M_i \xrightarrow{\tau_W^+} M$ if and only if for every $x \in X$, $\liminf_i d(x, M_i) \geq d(x, M)$ (i.e., for every $0 < \varepsilon < \varepsilon'$ with $S(x, \varepsilon') \cap M = \emptyset$, there is $i_0 \in I$ so that for every $i \in I$, with $i \geq i_0$, we have $S(x, \varepsilon) \cap M_i = \emptyset$).

- (ii) $M_i \xrightarrow{\tau_W^-} M$ if and only if for every $x \in X$, $\limsup_i d(x, M_i) \leq d(x, M)$ (i.e., for every $D \in \tau_d$ with $D \cap M \neq \emptyset$, there is $i_0 \in I$ so that for every $i \in I$, with $i \geq i_0$ we have $D \cap M_i \neq \emptyset$).

Remark 2.4 [14]

- (i) If (X, d) is a complete, separable metric space, then $\mathcal{P}_f(X)$ with the Wijsman topology is a Polish space (Beer [7]). Moreover, the space $(\mathcal{P}_f(X), \tau_W)$ is Polish if and only if (X, d) is Polish.
- (ii) $(\mathcal{P}_f(X), \tau_W)$ is a Tychonoff space. (X, d) is separable if and only if $\mathcal{P}_f(X)$ is either metrizable, first-countable or second-countable. The dependence of the Wijsman topology on the metric d is quite strong. Even if two metrics are uniformly equivalent, they may generate different Wijsman topologies. Necessary and sufficient conditions for two metrics to induce the same Wijsman topology have been found.

Hausdorff topology

In recent years, due to the development of computational graphics (for instance, in the automatic recognition of figures problems), it was necessary to measure accurately the matching, i.e., to calculate the distance between two sets of points. This led to the need to operate with an acceptable distance, which has to satisfy the first condition in the definition of a distance: the distance is zero if and only if the overlap is perfect. An appropriate metric in these issues is the Hausdorff metric on which we will refer in the following and which, roughly speaking, measures the degree of overlap of two compact sets.

Let $M, N \in \mathcal{P}_f(X)$. The *Hausdorff–Pompeiu pseudometric* h on $\mathcal{P}_f(X)$ is the “greatest” of all distances from any point in one of these two sets, to the nearest point from the other set, so, it is defined by

$$(*) \quad h(M, N) = \max\{e(M, N), e(N, M)\},$$

where $e(M, N) = \sup_{x \in M} d(x, N)$ is the *excess* of M over N and $d(x, N) = \inf_{y \in N} d(x, y)$ is the distance from x to N (with respect to the metric d).

For instance, the Cantor set $\mathcal{C} \in \mathcal{P}_k(\mathbb{R})$ and its “steps” $I_n \in \mathcal{P}_k(\mathbb{R}), \forall n \in \mathbb{N}$ (Kunze et al. [21]).

The topology induced by the Hausdorff pseudometric h is called the *Hausdorff hypertopology* τ_H on $\mathcal{P}_f(X)$.

On $\mathcal{P}_{bf}(X)$, h becomes a veritable metric. If, in addition, X is complete, then the same is $\mathcal{P}_f(X)$ (Hu and Papageorgiou [20]).



We observe that $e(N, M) = h(M, N)$, for every $M, N \in \mathcal{P}_f(X)$, with $M \subseteq N$. Also, $e(M, N) \leq e(M, P)$, for every $M, N, P \in \mathcal{P}_f(X)$, with $P \subseteq N$ and

$e(M, P) \leq e(N, P)$, for every $M, N, P \in \mathcal{P}_f(X)$, with $M \subseteq N$.

Generally, even if $M, N \in \mathcal{P}_k(X)$, then $e(M, N) \neq e(N, M)$.

If $M \in \mathcal{P}_f(X)$ and $\varepsilon > 0$ is arbitrary, but fixed, we consider the ε -dilatation of the set M

$$S(M, \varepsilon) = \{x \in X; \exists m \in M, d(x, m) < \varepsilon\}$$

$$\left(= \bigcup_{m \in M} \{x \in X; m \in M, d(x, m) < \varepsilon\} \right).$$

Obviously, $M \subseteq S(M, \varepsilon)$.

Since $h(M, N) < \varepsilon$ iff $M \subset S(N, \varepsilon)$ and $N \subset S(M, \varepsilon)$, we have the following equivalent expression for $h(M, N)$:

$$(**) \quad h(M, N) = \inf\{\varepsilon > 0, M \subset S(N, \varepsilon), N \subset S(M, \varepsilon)\}$$

($h(M, N)$ is the “smallest” $\varepsilon > 0$ which permits the ε -dilatation of M to cover N and the ε -dilatation of N to cover M).

In other words, $\tau_H = \tau_H^+ \cup \tau_H^-$, where τ_H^+ (upper Hausdorff topology), respectively, τ_H^- (lower Hausdorff topology) has as a base, the family $\{U^+(M, \varepsilon)\}_{\varepsilon > 0}$, where $U^+(M, \varepsilon) = \{N \in \mathcal{P}_f(X); N \subset S(M, \varepsilon)\}$, respectively, the family $\{U^-(M, \varepsilon)\}_{\varepsilon > 0}$, where $U^-(M, \varepsilon) = \{N \in \mathcal{P}_f(X); M \subset S(N, \varepsilon)\}$.

Another equivalent expression of the Hausdorff distance between two sets $M, N \in \mathcal{P}_f(X)$ is:

$$(***) \quad h(M, N) = \sup\{|d(x, N) - d(x, M)|; x \in X\}.$$

And this highlights the uniform aspect of the Hausdorff topology: it is the topology on $\mathcal{P}_f(X)$ of uniform convergence on X of the distance functionals $x \mapsto d(x, M)$, with $M \in \mathcal{P}_f(X)$.

Hausdorff topology is invariant with respect to uniformly equivalent metrics (Apreutesei [4]).

In the following, we list some properties of the Hausdorff metric:

Proposition 2.5

(I)

$$(i) \quad h(M_1 \cup M_2, N_1 \cup N_2) \leq \max\{h(M_1, N_1), h(M_2, N_2)\}, \forall M_1, M_2, N_1, N_2 \in \mathcal{P}_f(X);$$

If X is a Banach space, then:

(II)

$$(i) \quad h(\alpha M, \alpha N) = |\alpha| h(M, N), \forall \alpha \in \mathbb{R}, \forall M, N \in \mathcal{P}_f(X);$$

$$(ii) \quad h(M + P, N + P) \leq h(M, N), \forall M, N, P \in \mathcal{P}_f(X);$$

$$(ii') \quad h(\sum_{i=1}^p M_i, \sum_{i=1}^p N_i) \leq \sum_{i=1}^p h(M_i, N_i), \forall M_i, N_i \in \mathcal{P}_f(X)$$

(where $M + N = \{m + n; m \in M, n \in N\}$).

If, particularly, $X = \mathbb{R}$, and $a, b, c, d \in \mathbb{R}$, with $a < b, c < d$, then

$$h([a, b], [c, d]) = \max\{|a - c|, |b - d|\}.$$

Remark 2.6 [21] Hausdorff metric has some interesting characteristics:

(i) It is possible for a sequence of finite sets to converge to an uncountable set:

$$\forall n \geq 1, \text{ let be } M_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$$

$$(\subset [0, 1]) \text{ (all of them are finite sets).}$$

Since $h(M_n, [0, 1]) = \frac{1}{2n}$, then $M_n \xrightarrow{h} [0, 1]$ (in $\mathcal{P}_k(\mathbb{R})$), but $[0, 1]$ is uncountable.

(ii) Adding or removing a single point often influences the Hausdorff distance between two (compact) sets: if $M = [0, 1]$ and $N = [0, 1] \cup \{x\}$, where $x \notin [0, 1]$, then $h(M, N) = \max\{-x, x - 1\}$ (so, it is a function of x).

(iii) $I_n \xrightarrow{h} \mathcal{C}$ and $J_n \xrightarrow{h} \mathcal{C}$ where $(I_n)_{n \in \mathbb{N}}$ (respectively, $(J_n)_{n \in \mathbb{N}}$) are the steps in the construction of Cantor set \mathcal{C} .

Remark 2.7 [4, 14], [38, Ch. 1], [39, Ch. 8]

(i) If the pointwise convergence of Wijsman convergence is replaced by uniform convergence (uniformly in x), then one obtains Hausdorff convergence induced by the Hausdorff pseudometric. Generally, Hausdorff topology τ_H is finer than Wijsman topology τ_W . Hausdorff and Wijsman topologies on $\mathcal{P}_f(X)$ coincide if and only if (X, d) is totally bounded.

(ii) If X is a real normed space, then Hausdorff topology, Vietoris topology and Wijsman topology are equivalent on the class of monotone sequences of subsets of $\mathcal{P}_k(X)$.

Remark 2.8 Hausdorff metric on $\mathcal{P}_k(X)$ is an essential tool in the study of fractals and their generalizations: hyperfractals, multifractals and superfractals—[2, 3, 21].



Barnsley [6] calls the space $(\mathcal{P}_k(X), h)$, the *life space of fractal*. Recently, Banach and Novosad [5] proposed a fractal approach using Vietoris topology (in a more general setting than the one used for the Hausdorff topology).

Regular set multifunctions

Suppose that T is a locally compact, Hausdorff space, \mathcal{C} a ring of subsets of T and X a real normed space. Usually, it is assumed that \mathcal{C} is \mathcal{B}_0 (\mathcal{B}'_0 , respectively)—the Baire δ -ring (σ -ring, respectively) generated by compact sets, which are G_δ (i.e., countable intersections of open sets) or \mathcal{C} is \mathcal{B} (\mathcal{B}' , respectively)—the Borel δ -ring (σ -ring, respectively) generated by the compact sets of T .

$\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}'$, $\mathcal{B}_0 \subset \mathcal{B}'_0$. If T is metrisable or if it has a countable base, then any compact set $K \subset T$ is G_δ . In this case $\mathcal{B}_0 = \mathcal{B}$ (Dinculeanu [9, Ch. III, p. 187]) so $\mathcal{B}'_0 = \mathcal{B}'$.

By \mathcal{K} we denote the family of all compact subsets of T and by \mathcal{D} the family of all open subsets of T .

Regularity can be considered as a property of continuity with respect to a topology on $\mathcal{P}(T)$ (Dinculeanu [9, Ch. III, p. 197]):

For every $K \in \mathcal{K}$ and every $D \in \mathcal{D}$, with $K \subset D$, we denote $\mathcal{I}(K, D) = \{A \subset T / K \subset A \subset D\}$.

Since $\mathcal{I}(K, D) \cap \mathcal{I}(K', D') = \mathcal{I}(K \cup K', D \cap D')$, for every $\mathcal{I}(K, D), \mathcal{I}(K', D')$, the family $\{\mathcal{I}(K, D)\}_{\substack{K \in \mathcal{K} \\ D \in \mathcal{D}}}$ is a base of a topology $\tilde{\tau}$ on $\mathcal{P}(T)$. $\tilde{\tau}$ also denotes the topology induced on any subfamily $\mathcal{S} \subset \mathcal{P}(T)$ of subsets of T .

By $\tilde{\tau}_l$ ($\tilde{\tau}_r$, respectively) we denote the topology induced on $\{\mathcal{I}(K)\}_{K \in \mathcal{K}} = \{\{A \subset T / K \subset A\}\}_{K \in \mathcal{K}}$ ($\{\mathcal{I}(D)\}_{D \in \mathcal{D}} = \{\{A \subset T / A \subset D\}\}_{D \in \mathcal{D}}$, respectively) (Dinculeanu [9, Ch. III, p. 197–198]).

Definition 3.1 A class $\mathcal{F} \subset \mathcal{P}(T)$ is *dense* in $\mathcal{P}(T)$ with respect to the topology induced by $\tilde{\tau}$ if for every $K \in \mathcal{K}$ and every $D \in \mathcal{D}$, with $K \subset D$, there is $A \in \mathcal{F}$ such that $K \subset A \subset D$.

Since T is locally compact, the following statements can be easily verified (Dinculeanu [9, Ch. III, p. 197]):

Remark 3.2

- (1) $\mathcal{B}_0, \mathcal{B}, \mathcal{B}'_0, \mathcal{B}'$ are dense in $\mathcal{P}(T)$ with respect to the topology induced by $\tilde{\tau}$.
- (2)
 - (i) For every $A \in \mathcal{C}$, there exists $D \in \mathcal{D} \cap \mathcal{C}$ so that $A \subset D$.
 - (ii) If \mathcal{C} is \mathcal{B} or \mathcal{B}' , then for every $A \in \mathcal{C}$, there exist $K \in \mathcal{K} \cap \mathcal{C}$ and $D \in \mathcal{D} \cap \mathcal{C}$ so that $K \subset A \subset D$.

Let $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ be an arbitrary set multifunction.

Definition 3.3 μ is said to be *monotone* or *fuzzy* (with respect to the inclusion of sets) if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \subseteq B$.

Example 3.4 (of monotone set multifunctions)

- (i) Let \mathcal{C} be a ring of subsets of an abstract space T , $m : \mathcal{C} \rightarrow \mathbb{R}_+$ a finitely additive set function and $\mu : \mathcal{C} \rightarrow \mathcal{P}_{bf}(\mathbb{R})$ the set multifunction defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} [-m(A), m(A)], & \text{if } m(A) \leq 1 \\ [-m(A), 1], & \text{if } m(A) > 1 \end{cases}.$$

We easily observe that μ is monotone and $|\mu(A)| = m(A)$, for every $A \in \mathcal{C}$.

- (ii) Let $v_1, \dots, v_p : \mathcal{C} \rightarrow \mathbb{R}_+$, be p finitely additive set functions, where \mathcal{C} is a ring of subsets of an abstract space T . We consider the set multifunction $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(\mathbb{R})$, defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \{v_1(A), v_2(A), \dots, v_p(A)\}.$$

Then the set multifunction $\mu^\vee : \mathcal{C} \rightarrow \mathcal{P}_f(\mathbb{R})$, defined for every $A \in \mathcal{C}$ by:

$$\mu^\vee(A) = \overline{\bigcup_{\substack{B \subset A, \\ B \in \mathcal{C}}} \mu(B)}$$

is monotone.

In what follows, let $\mu : (\mathcal{C}, \tau_1) \rightarrow (\mathcal{P}_f(X), \tau_2)$ be a monotone set multifunction, where $\tau_1 \in \{\tilde{\tau}, \tilde{\tau}_l, \tilde{\tau}_r\}$ and $\tau_2 \in \{\tau_H, \tau_W, \tau_V\}$.

Let also be $\mathcal{B}_1 \in \{\{\mathcal{I}(K, D)\}_{\substack{K \in \mathcal{K} \\ D \in \mathcal{D}}}, \{\mathcal{I}(K)\}_{K \in \mathcal{K}}, \{\mathcal{I}(D)\}_{D \in \mathcal{D}}\}$, respectively, \mathcal{B}_2 , bases for τ_1 , respectively, τ_2 (as discussed in “[Hit-and-miss hypertopologies: an overview](#)”).

$\tau_2 = \tau_2^+ \cup \tau_2^-$, where $\tau_2^+ \in \{\tau_H^+, \tau_V^+, \tau_W^+\}$ and $\tau_2^- \in \{\tau_H^-, \tau_V^-, \tau_W^-\}$.

In a unifying way,

Definition 3.5 A set $A \in \mathcal{C}$ is said to be (τ_2) -regular if $\mu : (\mathcal{C}, \tau_1) \rightarrow (\mathcal{P}_f(X), \tau_2)$ is continuous at A , that is, for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exists $\tilde{\mathcal{V}} \in \mathcal{B}_1$ so that $\mu(\tilde{\mathcal{V}} \cap \mathcal{C}) \subset \mathcal{V}$ (or, equivalently, for every $(A_i)_{i \in I}, A \subset \mathcal{C}$, with $A_i \xrightarrow{\tau_1} A$, it results $\mu(A_i) \xrightarrow{\tau_2} \mu(A)$).

When τ_1 is $\tilde{\tau}$, or $\tilde{\tau}_l$ or $\tilde{\tau}_r$, respectively, we get the notions of (τ_2) -regularity, (τ_2) - R_l -regularity (inner regularity) or (τ_2) - R_r -regularity (outer regularity).

Precisely, we have:



Proposition 3.6 A is:

- (i) *regular* iff for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exist $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset D$, we have $\mu(B) \in \mathcal{V}$;
- (ii) R_l -regular iff for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exists $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we have $\mu(B) \in \mathcal{V}$;
- (iii) R_r -regular iff for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exists $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ so that for every $B \in \mathcal{C}$, with $A \subset B \subset D$, we have $\mu(B) \in \mathcal{V}$.

Remark 3.7 Every $K \in \mathcal{K}$ is R_l -regular and every $D \in \mathcal{D}$ is R_r -regular.

The following results can be proved using the above definitions:

Proposition 3.8

- (i) A set A is (τ_2) -regular if and only if it is (τ_2) - R_l -regular and (τ_2) - R_r -regular.
- (ii) A is (τ_2) -regular (R_l -regular or R_r -regular, respectively) if and only if it is (τ_2^+) - and (τ_2^-) -regular (R_l -regular or R_r -regular, respectively).

Theorem 3.9 Suppose $\mu_1, \mu_2 : \mathcal{C} \rightarrow (\mathcal{P}_f(X), \tau_2)$ are two monotone set multifunctions.

- (i) If μ_1, μ_2 are R_l -regular, then $\mu_1 = \mu_2$ on \mathcal{C} if and only if $\mu_1 = \mu_2$ on $\mathcal{K} \cap \mathcal{C}$;
- (ii) If μ_1, μ_2 are R_r -regular, then $\mu_1 = \mu_2$ on \mathcal{C} if and only if $\mu_1 = \mu_2$ on $\mathcal{D} \cap \mathcal{C}$.

Remark 3.10 For $\tau_2 = \tau_H, \tau_W$ or τ_V , respectively, we particularly get the notions of regularity as we defined and studied in [12–14]. For instance, if $\tau_2 = \tau_H$, then, by its monotonicity, μ is (in the sense of [12]):

- (i) *regular* if for every $\varepsilon > 0$, there are $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ so that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$.
- (ii) R_l -regular if for every $\varepsilon > 0$, there exists $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ so that $h(\mu(A), \mu(B)) = e(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset A$.
- (iii) R_r -regular if for every $\varepsilon > 0$, there exists $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ such that $h(\mu(A), \mu(B)) = e(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

In fact, one may easily observe that (in τ_H):

- (i) μ is regular iff for every $\varepsilon > 0$, there are $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ so that $e(\mu(D), \mu(K)) < \varepsilon$;
- (ii) μ is R_l -regular iff for every $\varepsilon > 0$, there is $K \in \mathcal{K} \cap \mathcal{C}$, $K \subset A$ so that $e(\mu(A), \mu(K)) < \varepsilon$;
- (iii) μ is R_r -regular iff for every $\varepsilon > 0$, there is $D \in \mathcal{D} \cap \mathcal{C}$, $D \supset A$ so that $e(\mu(D), \mu(A)) < \varepsilon$,

that is, in each case, we find an alternative expression of regularity as an approximation property.

Regularization by sets of functions of ε -approximation-type scale: physical correspondences with hit-and-miss topologies

In this section, analogously to our considerations from the previous section concerning regularity as an approximation property, we now study physical regularizations. Precisely, as we shall see, generally, the “reduction” of the complex dimensions to their real part requires the regularization by sets of functions of ε -approximation-type scale, while the “reduction” to their imaginary part requires regularization with “known” sets, that is, sets for which we have some informations.

We consider a fractal function $f(x)$, with $x \in [a, b]$ (for instance, one of the trajectory’s equation) and the sequence of the variable x values:

$$x_a = x_0, x_1 = x_0 + \varepsilon, \dots, x_k = x_0 + k\varepsilon, \dots, x_n = x_0 + n\varepsilon = x_b. \quad (1)$$

By $f(x, \varepsilon)$, we denote the fractured line connecting the points $f(x_0), \dots, f(x_k), \dots, f(x_n)$.

This line will be considered as an approximation which is different from the one used before. We shall say that $f(x, \varepsilon)$ is an ε -approximation scale.

Now, we consider the $\bar{\varepsilon}$ -approximation scale $f(x, \bar{\varepsilon})$ of the same function. When we study a fractal phenomenon by approximation, because $f(x)$ is similar almost everywhere, then, if ε and $\bar{\varepsilon}$ are small enough, the two approximations $f(x, \varepsilon)$ and $f(x, \bar{\varepsilon})$ must lead to the same results. If we compare the two cases, then to an infinitesimal increase $d\varepsilon$ of ε , it corresponds an increase $d\bar{\varepsilon}$ of $\bar{\varepsilon}$, if the scale is dilated.

In this case, $\frac{d\varepsilon}{\varepsilon} = \frac{d\bar{\varepsilon}}{\bar{\varepsilon}}$, i.e.,

$$\frac{d\varepsilon}{\varepsilon} = d\rho \quad (2)$$

is the ratio of the scale $\varepsilon + d\varepsilon$ and $d\varepsilon$ must be preserved.

Then, we can consider the infinitesimal transformation of the scale as

$$\varepsilon' = \varepsilon + d\varepsilon = \varepsilon + \varepsilon d\rho. \quad (3)$$

By such transformation, it results in the case of the function $f(x, \varepsilon)$:

$$f(x, \varepsilon') = f(x, \varepsilon + \varepsilon d\rho), \quad (4)$$

respectively, if we stop after the first approximation,

$$f(x, \varepsilon') = f(x, \varepsilon) + \frac{\partial f}{\partial \varepsilon}(\varepsilon' - \varepsilon), \quad (5)$$

i.e.,

$$f(x, \varepsilon') = f(x, \varepsilon) + \frac{\partial f}{\partial \varepsilon} \varepsilon d\rho. \quad (6)$$

We note that, for arbitrary, but fixed ε_0 ,

$$\frac{\partial \ln \frac{\varepsilon}{\varepsilon_0}}{\partial \varepsilon} = \frac{\partial (\ln \varepsilon - \ln \varepsilon_0)}{\partial \varepsilon} = \frac{1}{\varepsilon}, \quad (7)$$

so Eq. 5 becomes

$$f(x, \varepsilon') = f(x, \varepsilon) + \frac{\partial f(x, \varepsilon)}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} d\rho. \quad (8)$$

Finally, we get

$$f(x, \varepsilon') = \left(1 + \frac{\partial}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} d\rho\right) f(x, \varepsilon). \quad (9)$$

The operator

$$\tilde{D} = \frac{\partial}{\partial \ln \frac{\varepsilon}{\varepsilon_0}} \quad (10)$$

is called the *dilatation operator*.

The above relation shows that the intrinsic variable of the resolution is not ε , but $\ln \frac{\varepsilon}{\varepsilon_0}$.

On the other hand, simultaneous invariance with respect to both space–time coordinates and the resolution scale induces general scale relativity theory (SRT) [32, 33]. These theories are more general than Einstein's general relativity theory, being invariant with respect to the generalized Poincaré group (standard Poincaré group and dilatation group) [32, 33].

Basically, we discuss various physical theories built on manifolds of fractal space–time and they all turn out to be reducible to one of the following classes:

- (i) SRT [35, 36] and its possible extensions [34]. It is considered that the microparticles motion takes place on continuous but non-differentiable curves. In such context, regularization works using sets of functions of ε -approximation-type scale.
- (ii) Transition in which to each point of the motion trajectory, a transfinite set is assigned (in particular, a Cantor-type set—see the El Naschie [34] $\varepsilon^{(\infty)}$ model of space–time), to mimic the

continuous (the trans-physics). In such context, the regularization of “vague” sets by known sets works.

- (iii) Fractal string theories containing simultaneously relativity and trans-physics [19, 37].

The reduction of the complex dimensions to their real part is equivalent to Scale Relativity-Type theories, while reducing them to the imaginary part of their complex dimensions generates trans-physics. In such context, the simultaneous regularization by sets of functions of ε -approximation-type scale and also by “known” sets works. The “reduction” of the complex dimensions to their real part requires the regularization by sets of functions of ε -approximation-type scale, while the “reduction” to their imaginary part requires regularization with “known” sets.

Dynamical systems behaviors are collective phenomena emerging out of many segregated components. Most of these systems are collective (that is, set-valued) dynamics of many units of individual systems, whence the need of a (hyper)fractal topological treatment of such collective dynamics.

We consider that the particle of a complex system moves on continuous, but non-differentiable curves (fractal curves). Once accepted such a hypothesis, some consequences of non-differentiability by SRT are evident [35, 36].

For instance, physical quantities that describe the complex system are fractal functions, i.e., functions depending both on spatial coordinates and time as well as on the scale resolution $\frac{\delta t}{\tau}$.

In classical physics, the physical quantities describing the dynamics of a complex system are continuous, but differentiable functions depending only on spatial coordinates and time.

Since [1, 30, 31, 35, 36], two representations are complementary: the formalism of the fractal hydrodynamics (at the continuum level), and the one of the Schrödinger-type theory (at the discontinuum level). Moreover, the chaoticity, either through turbulence in the fractal hydrodynamic approach, either through stochasticization in the Schrödinger-type approach, is generated only by the non-differentiability of the movement trajectories in a fractal space.

Conclusions

In this paper, we intend to present a unifying mathematical–physical perspective concerning the relationships, interpretations and similitudes existing among fractality, regularity and several hit-and-miss hypertopologies. We intend to continue the study of regularity in hypertopologies viewed in the context of domain theory (in correlation with [10, 17]). We are also interested in developing a neural network fractal theory using Wijsman topology (its



pointwise character seems to characterize some properties better than the Hausdorff topology induced by the Hausdorff–Pompeiu metric (which has a uniform character).

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